7 Global Linkages and Economic Growth

$$Y_t = F(K_t, E_t L_t), \tag{1}$$

$$Y_t - C_t = S_t = sF(K_t, E_t L_t). \tag{2}$$

$$K_{t+1} - K_t = sF(K_t, E_t L_t) - \delta K_t, \tag{3}$$

$$E_{t+1} = (1+g)E_t, \tag{4}$$

$$L_{t+1} = (1+n)L_t. \tag{5}$$

$$k_t^{\mathrm{E}} = K_t/E_t L_t. \tag{6}$$

$$\frac{K_{t+1}}{E_t L_t} - \frac{K_t}{E_t L_t} = \frac{sF(K_t, E_t L_t)}{E_t L_t} - \delta \frac{K_t}{E_t L_t} \tag{7}$$

$$\frac{K_{t+1}}{E_t L_t} = \frac{K_{t+1}}{E_{t+1} L_{t+1}} \cdot \frac{E_{t+1} L_{t+1}}{E_t L_t} = k^{\mathrm{E}}_{t+1}(1+z)$$

$$k^{\mathrm{E}}_{t+1} - k^{\mathrm{E}}_t = \frac{1}{1+z}\left[sf(k^{\mathrm{E}}_t) - (z+\delta)k^{\mathrm{E}}_t\right], \qquad (8)$$

$$sf(\bar{k}^{\mathrm{E}}) = (z+\delta)\bar{k}^{\mathrm{E}}. \qquad (9)$$

$$\bar{k}^{\mathrm{E}} = \left(\frac{s}{z+\delta}\right)^{\frac{1}{1-\alpha}}. \qquad (10)$$

$$\frac{Y_t}{L_t} = E_t \left(\frac{s}{z + \delta} \right)^{\frac{\alpha}{1-\alpha}}$$

$$\log \frac{Y_t}{L_t} = \log E_0 + t \log(1 + g) + \frac{\alpha}{1 - \alpha} \log s - \frac{\alpha}{1 - \alpha} \log(z + \delta)$$

$$\log \frac{Y_t}{L_t} = \log E_0 + gt + \frac{\alpha}{1 - \alpha} \log s - \frac{\alpha}{1 - \alpha} \log(n + g + \delta). \tag{11}$$

$$\log \frac{Y_t}{L_t} = 5.48 + 1.42 \log s - 1.97 \log(n + g + \delta), \quad R^2 = 0.59.$$

$$\begin{matrix} (1.59) & (0.14) & & (0.56) \end{matrix} \tag{12}$$

$$Y_t = K_t^{\alpha} H_t^{\phi} \left(E_t L_t\right)^{1-\alpha-\phi},$$

$$y_t^{\mathrm{E}} = (k_t^{\mathrm{E}})^{\alpha} (h_t^{\mathrm{E}})^{\phi}, \tag{13}$$

$$H_{t+1} - H_t = s_{\mathrm{H}}[K_t^{\alpha} H_t^{\phi} (E_t L_t)^{1-\alpha-\phi}] - \delta H_t$$

$$h^{\mathrm{E}}_{t+1} - h^{\mathrm{E}}_{t} = \frac{1}{1+z}\{s_{\mathrm{H}}[(k^{\mathrm{E}})^{\alpha}(h^{\mathrm{E}})^{\phi}] - (z+\delta)h^{\mathrm{E}}_{t}\}. \quad (14)$$

$$K_{t+1} - K_t = s_{\mathrm{K}}[K_t^{\alpha} H_t^{\phi}(E_t L_t)^{1-\alpha-\phi}] - \delta K_t$$

$$k^{\mathrm{E}}_{t+1} - k^{\mathrm{E}}_{t} = \frac{1}{1+z}\{s_{\mathrm{K}}[(k^{\mathrm{E}})^{\alpha}(h^{\mathrm{E}})^{\phi}] - (z+\delta)k^{\mathrm{E}}_{t}\}. \quad (15)$$

$$\bar{k}^{\mathrm{E}} \equiv \left[\frac{(s_{\mathrm{K}})^{1-\phi}(s_{\mathrm{H}})^{\phi}}{z+\delta}\right]^{\frac{1}{1-\alpha-\phi}}, \quad (16)$$

$$\bar{h}^{\mathrm{E}} \equiv \left[\frac{(s_{\mathrm{H}})^{1-\alpha} (s_{\mathrm{K}})^{\alpha}}{z+\delta} \right]^{\frac{1}{1-\alpha-\phi}}. \tag{17}$$

$$\log \frac{Y_t}{L_t} = \log E_0 + gt + \frac{\alpha}{1-\alpha-\phi} \log s_{\mathrm{K}} + \frac{\phi}{1-\alpha-\phi} \log s_{\mathrm{H}}$$
$$- \log \frac{\alpha+\phi}{1-\alpha-\phi}(n+g+\delta),$$

$$U_t = L_t \sum_{s=t}^{\infty} \beta^{s-t} (1+n)^{s-t} u\left(c_s\right), \tag{18}$$

$$K_{t+1} = K_t + F(K_t, E_t L_t) - C_t.$$

$$k_{t+1} - k_t = \frac{F(k_t, E_t) - c_t}{1+n} - \frac{nk_t}{1+n}, \tag{19}$$

$$u'(c_t) = [1 + F_K(k_{t+1}, E_{t+1})]\beta u'(c_{t+1}). \tag{20}$$

$$u\ (c) = \frac{c^{1-\frac{1}{\sigma}}}{1 - \frac{1}{\sigma}}, \tag{21}$$

$$c_{t+1}/c_t = \beta^{\sigma} \left[1 + F_K(k_{t+1}, E_{t+1})\right]^{\sigma} . \tag{22}$$

$$\beta(1+n)(1+g)^{(\sigma-1)/\sigma} < 1, \tag{23}$$

$$k_{t+1}^{\mathrm{E}} - k_t^{\mathrm{E}} = \frac{f(k_t^{\mathrm{E}}) - c_t^{\mathrm{E}}}{1+z} - \frac{z}{1+z}k_t^{\mathrm{E}}, \tag{8'}$$

$$\frac{c_{t+1}^{\mathrm{E}}}{c_t^{\mathrm{E}}} = \frac{\beta^{\sigma}[1+f'(k_{t+1}^{\mathrm{E}})]^{\sigma}}{1+g}, \tag{24}$$

$$f'(\bar{k}^{\mathrm{E}}) = \frac{(1+g)^{1/\sigma}}{\beta} - 1. \tag{25}$$

$$\bar{c}^{\mathrm{E}} = f(\bar{k}^{\mathrm{E}}) - z\bar{k}^{\mathrm{E}}. \tag{26}$$

$$\begin{aligned} U_t &= L_t \sum_{s=t}^{\infty} \left(\frac{\beta}{1+n}\right)^{s-t} (1+n)^{s-t} u(c_s) \\ &= L_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \end{aligned} \tag{27}$$

$$f'(\bar{k}^{\mathrm{E}}) = \frac{(1+g)^{1/\sigma}}{\beta/\,(1+n)} - 1 \tag{28}$$

$$U_t^v = \sum_{s=t}^{\infty} \beta^{s-t} \log(c_s^v) \tag{29}$$

$$k_{t+1}^v = (1 + r_t)k_t^v + w_t - c_t^v, \tag{30}$$

$$\frac{c_{t+1}^v}{c_t^v} = (1 + r_{t+1})\beta. \tag{31}$$

$$x_t = \frac{x_t^0 + nx_t^1 + n(1+n)x_t^2 + \ldots + n(1+n)^{t-1}x_t^t}{(1+n)^t}.$$

$$k_{t+1} - k_t = \frac{f(k_t) - c_t}{1+n} - \frac{nk_t}{1+n}. \tag{32}$$

$$\frac{c_{t+1}^0 + nc_{t+1}^1 + \ldots + n(1+n)^{t-1}c_{t+1}^t}{(1+n)^t}$$

$$= (1 + r_{t+1})\beta c_t$$

$$c_{t+1} = (1 + r_{t+1})\beta c_t - n\left(c_{t+1} - c_{t+1}^{t+1}\right).$$

$$c_{t+1} - c_{t+1}^{t+1} = (1-\beta)(1 + r_{t+1})k_{t+1}.$$

$$c_{t+1} = (1 + r_{t+1})\beta c_t - n(1 - \beta)(1 + r_{t+1})k_{t+1}.$$

$$c_{t+1} = [1 + f'(k_{t+1})][\beta c_t - n(1 - \beta)k_{t+1}]. \tag{33}$$

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \log c_s^{\mathrm{N}}, \tag{34}$$

$$h_{t+1}^{\mathrm{N}} = (1 + w_{\mathrm{s},t} - \delta)h_t^{\mathrm{N}} + w_t - c_t^{\mathrm{N}}, \tag{35}$$

$$\frac{c_{t+1}^{\mathrm{N}}}{c_t^{\mathrm{N}}} = (1 + w_{\mathrm{s},t+1} - \delta)\beta \tag{36}$$

$$w_t = f(h_t) - h_t f'(h_t), \tag{37}$$

$$w_{\mathrm{s},t} = f'(h_t), \tag{38}$$

$$h_{t+1} - h_t = f(h_t) - \delta h_t - c_t. \tag{39}$$

$$\frac{c_{t+1}}{c_t} = [1 + f'(h_{t+1}) - \delta]\beta. \tag{40}$$

$$\bar{h} = (f')^{-1}\left(\frac{1-\beta}{\beta} + \delta\right).$$

$$c = \frac{Nc^{\mathrm{N}} + Mc^{\mathrm{M}}}{N+M}, \tag{41}$$

$$F(N\bar{h}, N+M) - F(N\bar{h}, N) - MF_L(N\bar{h}, N+M).$$

$$\frac{F(N\bar{h}, N+M) - F(N\bar{h}, N)}{M} > F_L(N\bar{h}, N+M).$$

$$\log\left(\frac{Y_{1990}}{L_{1990}}\right) - \log\left(\frac{Y_{1950}}{L_{1950}}\right) = \underset{(0.54)}{6.47} - \underset{(0.06)}{0.58} \log\left(\frac{Y_{1950}}{L_{1950}}\right), \quad R^2 = 0.83.$$

$$\log\left(\frac{Y_{1979}}{N_{1979}}\right) - \log\left(\frac{Y_{1870}}{N_{1870}}\right) = 8.46 - \underset{(0.09)}{1.00} \log\left(\frac{Y_{1870}}{N_{1870}}\right), \quad R^2 = 0.88.$$

$$\bar{k}^{\mathrm{E}} = \left[\frac{(1-\tau)\,\alpha}{r^{\mathrm{W}}}\right]^{1/(1-\alpha)}. \tag{42}$$

$$k_{t+1}^{\mathrm{E}} - k_t^{\mathrm{E}} = \frac{s(k_t^{\mathrm{E}})^{\alpha}}{1+z} - \frac{z+\delta}{1+z} k_t^{\mathrm{E}}, \tag{43}$$

$$\bar{k}^{\mathrm{E}} = \left(\frac{s}{z+\delta} \right)^{\frac{1}{1-\alpha}}.$$

$$k_{t+1}^{\mathrm{E}} - \bar{k}^{\mathrm{E}} = \mu (k_t^{\mathrm{E}} - \bar{k}^{\mathrm{E}}), \tag{44}$$

$$\mu \equiv \left[1 + \frac{s\alpha(\bar{k}^{\mathrm{E}})^{\alpha-1}}{1+z} - \frac{z+\delta}{1+z}\right]$$

$$= \frac{1 + \alpha z + (\alpha - 1)\delta}{1+z} \tag{45}$$

$$\mu = \frac{1 + (1/3)\left[(1.02)(1.01) - 1\right] - (2/3)(0.03)}{(1.02)(1.01)} \approx 0.96$$

$$Y_t = K_t^{\alpha} H_t^{\phi} \, (E_t L_t)^{1-\alpha-\phi},$$

$$y_t^{\mathrm{E}} = (k_t^{\mathrm{E}})^{\alpha} (h_t^{\mathrm{E}})^{\phi}.$$

$$r = \alpha (k_t^{\mathrm{E}})^{\alpha - 1} (h_t^{\mathrm{E}})^{\phi}.$$

$$k_t^{\mathrm{E}} = \frac{\alpha y_t^{\mathrm{E}}}{r}. \tag{46}$$

$$y_t^{\mathrm{E}} = \chi (h_t^{\mathrm{E}})^{\nu}, \tag{47}$$

$$\nu \equiv \frac{\phi}{1 - \alpha},$$

$$\chi \equiv \left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}}.$$

$$H_{t+1} - H_t + K_{t+1} - K_t + B_{t+1} - B_t$$
$$= Y_t + rB_t - C_t - \delta H_t$$

$$B_t = -K_t.$$

$$H_{t+1} - H_t = Y_t - rK_t - C_t - \delta H_t$$

$$H_{t+1} - H_t = s(Y_t - rK_t).$$

$$h^{\mathrm{E}}_{t+1} - h^{\mathrm{E}}_t = \frac{s'(h^{\mathrm{E}}_t)^{\nu}}{1+z} - \frac{z+\delta}{1+z} h^{\mathrm{E}}_t, \tag{48}$$

$$s' \equiv s(1-\alpha)\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}}.$$

$$\bar{h}^{\mathrm{E}} = \left(\frac{s'}{z+\delta}\right)^{1/(1-\nu)}, \tag{49}$$

$$h_{t+1}^{\mathrm{E}} - \bar{h}^{\mathrm{E}} = \mu'(h_t^{\mathrm{E}} - \bar{h}^{\mathrm{E}}), \tag{50}$$

$$\begin{aligned} \mu' &\equiv \left[1 + \frac{s'\nu(\bar{h}^{\mathrm{E}})^{\nu-1}}{1+z} - \frac{z+\delta}{1+z} \right] \\ &= \frac{1 + \nu z + (\nu - 1)\delta}{1+z} \end{aligned} \tag{51}$$

$$Y_t = (K_t^{\mathrm{G}})^{\phi} K^{\alpha} (E_t L_t)^{1-\alpha-\phi},$$

$$\Delta \mathsf{y}_t - \alpha \Delta \mathsf{k}_t - (1-\alpha)\Delta \mathsf{l}_t,$$

$$y_t = k_t^{\alpha}.$$

$$r^{\mathrm{D}} = f'(k) - \delta = \alpha k^{\alpha-1} - \delta. \tag{52}$$

$$k_{t+1} + b_{t+1} = w_t - c_t^{\mathrm{Y}}, \tag{53}$$

$$c^{\mathrm{O}}_{t+1} = (1 + r^{\mathrm{D}}_{t+1})k_{t+1} + (1 + r)b_{t+1} \tag{54}$$

$$b_{t+1} \geq -\eta w_t. \tag{55}$$

$$U_t = \log(c^{\mathrm{Y}}_t) + \beta \log(c^{\mathrm{O}}_{t+1})$$

$$c^{\mathrm{Y}}_t + \frac{c^{\mathrm{O}}_{t+1}}{1 + r^{\mathrm{D}}_{t+1}} = w_t - \frac{\left(r^{\mathrm{D}}_{t+1} - r\right)}{1 + r^{\mathrm{D}}_{t+1}} b_{t+1}. \tag{56}$$

$$s_t^{\mathrm{Y}} = w_t - c_t^{\mathrm{Y}} = \frac{\beta w_t}{1+\beta} - \frac{\left(r_{t+1}^{\mathrm{D}} - r\right) \eta w_t}{(1+\beta)\left(1 + r_{t+1}^{\mathrm{D}}\right)}.$$

$$k_{t+1} = s_t^{\mathrm{Y}} - b_{t+1} = \left[\frac{\beta(1+\eta)}{1+\beta} + \frac{(1+r)\eta}{(1+\beta)(1+r_{t+1}^{\mathrm{D}})}\right] w_t \tag{57}$$

$$k_{t+1} = (1-\alpha)\left[\frac{\beta(1+\eta)}{1+\beta} + \frac{(1+r)\eta}{(1+\beta)\alpha k_{t+1}^{\alpha-1}}\right] k_t^{\alpha}. \tag{58}$$

$$\bar{k}^{\mathrm{D}} = \left[\frac{\alpha\beta(1-\alpha)(1+\eta)}{\alpha(1+\beta) - \eta(1-\alpha)(1+r)}\right]^{\frac{1}{1-\alpha}}. \tag{59}$$

$$\frac{\beta\bar{w}}{1+\beta}+\eta\bar{w}<\bar{k}^{\mathrm{U}}, \tag{60}$$

$$\frac{\beta w_0}{1+\beta}+\eta w_0\geq\bar{k}^{\mathrm{U}},$$

$$U_t=\sum_{s=t}^{\infty}\beta^{s-t}\frac{c_s^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \qquad \sigma>0. \tag{61}$$

$$1 + r_{t+1} = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\frac{1}{\sigma}}, \tag{62}$$

$$y_t = Ak_t, \tag{63}$$

$$r_{t+1} = A. \tag{64}$$

$$c_t + i_t = y_t = Ak_t,$$

$$\frac{c_{t+1}}{c_t} = [\beta(1+A)]^\sigma = 1 + \bar{g}. \tag{65}$$

$$i_t = k_{t+1} - k_t = \bar{g}k_t = \frac{\bar{g}}{A}y_t,$$

$$c_t = \frac{A - \bar{g}}{A}y_t.$$

$$y_t^j = A\left(k_t^j\right)^\alpha k_t^{1-\alpha}, \tag{66}$$

$$\frac{\mathrm{d}y^j}{\mathrm{d}k^j} = \alpha A \left(\frac{k}{k^j}\right)^{1-\alpha} . \tag{67}$$

$$\frac{\mathrm{d}y}{\mathrm{d}k} = \alpha A = r. \tag{68}$$

$$\frac{c_{t+1}}{c_t} = \left[\beta(1 + \alpha A)\right]^{\sigma} = 1 + \bar{g}.$$

$$y_t = Ak_t,$$

$$U_t = \mathrm{E}_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \log C_s \right\}, \tag{69}$$

$$K_{t+1} = \left[x_t(1 + \tilde{r}_t) + (1 - x_t)(1 + r) \right] K_t - C_t, \tag{70}$$

$$1 = (1 + r)\beta \mathrm{E}_t \left\{ \frac{C_t}{C_{t+1}} \right\} \tag{71}$$

$$1 = \beta \mathrm{E}_t \left\{ (1 + \tilde{r}_{t+1}) \frac{C_t}{C_{t+1}} \right\}. \tag{72}$$

$$C_t = (1-\beta)\left[x_t(1+\tilde{r}_t) + (1-x_t)(1+r)\right] K_t \tag{73}$$

$$\mathrm{E}_t(\tilde{r}_{t+1}) - r \approx (1+r)\beta \mathrm{Cov}_t \left\{ \frac{C_{t+1}}{C_t} - 1, \tilde{r}_{t+1} - r \right\} \tag{74}$$

$$\frac{C_{t+1}}{C_t} - 1 = \beta\left[1 + r + x(\tilde{r}_{t+1} - r)\right] - 1, \tag{75}$$

$$\begin{aligned} &\mathrm{E}_t(\tilde{r}_{t+1} - r) \\ &\quad\approx (1+r)\beta \mathrm{Cov}_t \left\{\beta[1 + r + x(\tilde{r}_{t+1} - r)] - 1, \tilde{r}_{t+1} - r\right\} \\ &\quad= x(1+r)\beta^2 \mathrm{Var}_t(\tilde{r}_{t+1} - r). \end{aligned}$$

$$x = \frac{\mathrm{E}_t(\tilde{r}_{t+1} - r)}{\beta^2(1 + r)\mathrm{Var}_t(\tilde{r}_{t+1} - r)} \tag{76}$$

$$\mathrm{E}_t\left\{\frac{C_{t+1}}{C_t}\right\} = \frac{\left[\mathrm{E}_t(\tilde{r}_{t+1} - r)\right]^2}{\beta(1 + r)\mathrm{Var}_t(\tilde{r}_{t+1} - r)} + \beta(1 + r) \tag{77}$$

$$\mathrm{E}_t\left\{\frac{C^n_{t+1}}{C^n_t}\right\} = \frac{\left[\mathrm{E}_t(\tilde{r}^{\mathrm{w}}_{t+1} - r)\right]^2}{\beta(1 + r)\mathrm{Var}_t(\tilde{r}^{\mathrm{w}}_{t+1} - r)} + \beta(1 + r) \tag{78}$$

$$\begin{aligned} \mathsf{y}_t - \mathsf{y}_{t-1} = {} & \mathsf{a}_t - \mathsf{a}_{t-1} + \alpha(\mathsf{k}_t - \mathsf{k}_{t-1}) \\ & + (1 - \alpha)(\mathsf{l}_t - \mathsf{l}_{t-1}) \end{aligned}$$

$$Y_t = L_{Y,t}^{1-\alpha} \sum_{j=1}^{A_t} K_{j,t}^{\alpha}, \tag{79}$$

$$\frac{\partial Y}{\partial K_j} = \alpha L_Y^{1-\alpha} K_j^{\alpha-1} \tag{80}$$

$$\alpha L_Y^{1-\alpha} K_j^{\alpha-1} \mid_{K_j=0} \; = \; \infty.$$

$$A_{t+1} - A_t = \theta A_t L_{A,t}, \tag{81}$$

$$L = L_A + L_Y. \tag{82}$$

$$\max_{\{K_j\}} L_Y^{1-\alpha} \sum_{j=1}^{A_t} K_j^{\alpha} - \sum_{j=1}^{A_t} p_j K_j, \tag{83}$$

$$p_j = \alpha L_Y^{1-\alpha} K_j^{\alpha-1}. \tag{84}$$

$$\Pi_j = \frac{p_j K_j}{1+r} - K_j = \frac{\alpha L_Y^{1-\alpha} K_j^{\alpha}}{1+r} - K_j. \tag{85}$$

$$\bar{K} = \left(\frac{\alpha^2}{1+r} \right)^{1/(1-\alpha)} L_Y, \tag{86}$$

$$\bar{p} = \frac{1+r}{\alpha}. \tag{87}$$

$$\bar{\Pi} = \frac{\bar{p}\bar{K}}{1+r} - \bar{K} = \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{\alpha^2}{1+r} \right)^{\frac{1}{1-\alpha}} L_Y \tag{88}$$

$$\bar{p}_A = \sum_{s=t}^{\infty} \frac{\bar{\Pi}}{(1+r)^{s-t}} = \frac{(1+r)\bar{\Pi}}{r}. \tag{89}$$

$$\bar{g} = \frac{A_{t+1} - A_t}{A_t} = \theta \bar{L}_A. \tag{90}$$

$$\frac{\partial(\bar{p}_A \theta A L_A)}{\partial L_A} = \bar{p}_A \theta A = (1-\alpha) L_Y^{-\alpha} A \bar{K}^{\alpha} = \frac{\partial Y}{\partial L_Y} \tag{91}$$

$$\bar{L}_Y = \frac{r}{\theta \alpha}, \tag{92}$$

$$\bar{g} = \theta L - \frac{r}{\alpha}.$$

$$1 + \bar{g} = \left(\theta L + \frac{1+\alpha}{\alpha}\right) - \frac{1+r}{\alpha}. \tag{93}$$

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \frac{c_s^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},$$

$$1 + g = \frac{C_{t+1}}{C_t} = [(1+r)\beta]^{\sigma},$$

$$1 + r = \frac{1}{\beta}(1 + g)^{\frac{1}{\sigma}}. \tag{94}$$

$$\bar{r} = \frac{\alpha(1 + \theta L - \beta)}{1 + \alpha\beta}, \tag{95}$$

$$\bar{g} = \frac{\alpha\beta\theta L - (1 - \beta)}{1 + \alpha\beta}. \tag{96}$$

$$\theta L > \frac{1 - \beta}{\alpha\beta}$$

$$Y_t = L_{Y,t}^{1-\alpha} A_t K_t^{\alpha}$$

$$Y_t = C_t + A_{t+1} K_{t+1}.$$

$$\bar{g}^{\text{PLAN}} = \beta\theta L - (1 - \beta). \tag{97}$$

$$\bar{g} = \frac{\beta\theta L - (1 - \beta)}{1 + \beta},$$

$$A_{t+1} - A_t = \theta A_t L_t, \tag{98}$$

$$Y_t = A_t L_t^{1-\alpha}. \tag{99}$$

$$\bar{C}^{\mathrm{MIN}} = \frac{Y_t}{L_t}, \tag{100}$$

$$A_t = \bar{C}^{\mathrm{MIN}} L_t^{\alpha}. \tag{101}$$

$$L_{t+1}^{\alpha} - L_t^{\alpha} = \theta L_t^{1+\alpha},$$

$$\frac{L_{t+1}}{L_t} = (\theta L_t + 1)^{\frac{1}{\alpha}}. \tag{102}$$

$$Y_t = A_t F(Z_{Y,t}, L_{Y,t}),$$
$$X_t = A_t F(Z_{X,t}, L_{X,t}),$$

$$L_X + L_Y = \bar{L},$$

$$Z_X + Z_Y = \bar{Z}.$$

$$A_{t+1} - A_t = \theta X_t A_t.$$

$$U_t = \mathrm{E}_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \log C_s \right\}, \tag{103}$$

$$Y_t = A_t K_t^{\alpha}.$$

$$K_{t+1} = A_t K_t^{\alpha} - C_t. \tag{104}$$

$$\frac{1}{C_t} = \beta \mathrm{E}_t \left\{ \frac{1 + \tilde{r}_{t+1}}{C_{t+1}} \right\}. \tag{105}$$

$$\tilde{r}_{t+1} = \alpha A_{t+1} K_{t+1}^{\alpha-1} - 1, \tag{106}$$

$$1 = \beta \mathrm{E}_t \left\{ \alpha A_{t+1} K_{t+1}^{\alpha-1} \left(\frac{C_t}{C_{t+1}} \right) \right\}, \tag{107}$$

$$C_t = \omega A_t K_t^{\alpha} \tag{108}$$

$$\frac{1}{A_t K_t^{\alpha}} = \frac{\beta\alpha}{K_{t+1}},$$

$$\omega = 1 - \alpha\beta. \tag{109}$$

$$K_{t+1} = \alpha\beta A_t K_t^{\alpha}.$$

$$\mathsf{y}_t = \chi_0 + \alpha \mathsf{y}_{t-1} + \mathsf{a}_t, \tag{110}$$

$$\bar{\mathsf{y}} = \frac{\chi_0 + \bar{\mathsf{a}}}{1 - \alpha}. \tag{111}$$

$$\bar{\mathsf{y}} = \chi_0 + \alpha \bar{\mathsf{y}} + \bar{\mathsf{a}}$$

$$\mathsf{y}_t - \bar{\mathsf{y}} = \alpha(\mathsf{y}_{t-1} - \bar{\mathsf{y}}) + (\mathsf{a}_t - \bar{\mathsf{a}}), \tag{112}$$

$$\mathsf{y}_t - \bar{\mathsf{y}} = \sum_{s=1}^{t} \alpha^{t-s}(\mathsf{a}_s - \bar{\mathsf{a}}) + (\mathsf{y}_0 - \bar{\mathsf{y}})\alpha^t. \quad (113)$$

$$Y_t = A_t^{\mathrm{W}} \cdot A_t K_t{}^{\alpha}, \quad Y_t^* = A_t^{\mathrm{W}} \cdot A_t^* \left(K_t^*\right)^{\alpha}, \quad \alpha < 1$$

$$K_{t+1} + K_{t+1}^* = A_t^{\mathrm{W}} \left[A_t K_t^{\alpha} + A_t^*(K_t^*)^{\alpha}\right] - \left(C_t + C_t^*\right) \quad (114)$$

$$\frac{1}{C_t^i} = \beta \mathrm{E}_t \left\{\alpha A_{t+1}^{\mathrm{W}} \cdot A_{t+1}^j (K_{t+1}^j)^{\alpha-1} \left(\frac{1}{C_{t+1}^i}\right)\right\}$$

$$C_t = \kappa(1 - \alpha\beta)Y_t^{\mathrm{W}}, \tag{115}$$

$$K_{t+1} = \Psi\alpha\beta Y_t^{\mathrm{W}}, \tag{116}$$

$$\Psi \equiv \mathrm{E}_t \left\{ \frac{Y_{t+1}}{Y_{t+1}^{\mathrm{W}}} \right\} = \frac{K_{t+1}}{K_{t+1}^{\mathrm{W}}}.$$

$$U_t = \mathrm{E}_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \log C_s \right\},$$

$$K_{t+1} - K_t = Y_t - C_t. \tag{117}$$

$$Y_t = K_t^{\alpha} E_t{}^{1-\alpha}, \tag{118}$$

$$\frac{1}{C_t} = \beta \mathrm{E}_t \left\{ \frac{1 + \tilde{r}_{t+1}}{C_{t+1}} \right\}, \tag{119}$$

$$1 + \tilde{r}_{t+1} = 1 + \alpha \left(\frac{K_{t+1}}{E_{t+1}} \right)^{\alpha - 1}, \tag{120}$$

$$\frac{1}{\beta} = 1 + \bar{r}, \tag{121}$$

$$\frac{\bar{E}}{\bar{K}} = \left(\frac{1-\beta}{\beta\alpha}\right)^{\frac{1}{1-\alpha}}, \tag{122}$$

$$\frac{\bar{Y}}{\bar{K}} = \frac{1-\beta}{\beta\alpha}. \tag{123}$$

$$\frac{\bar{C}}{\bar{Y}} = 1. \tag{124}$$

$$\mathsf{y}_t = \alpha \mathsf{k}_t + (1-\alpha)\mathsf{e}_t. \tag{125}$$

$$\mathrm{d}K_{t+1} - \mathrm{d}K_t = \alpha \left(\frac{\bar{E}}{\bar{K}}\right)^{1-\alpha} \mathrm{d}K_t$$

$$+ (1-\alpha)\left(\frac{\bar{E}}{\bar{K}}\right)^{-\alpha} \mathrm{d}E_t - \mathrm{d}C_t$$

$$\frac{\mathrm{d}K_{t+1}}{\bar{K}} = \left[1 + \alpha\left(\frac{\bar{E}}{\bar{K}}\right)^{1-\alpha}\right]\frac{\mathrm{d}K_t}{\bar{K}}$$

$$+ (1-\alpha)\left(\frac{\bar{E}}{\bar{K}}\right)^{1-\alpha}\frac{\mathrm{d}E_t}{\bar{E}} - \frac{\bar{C}}{\bar{K}} \cdot \frac{\mathrm{d}C_t}{\bar{C}}$$

$$\mathsf{k}_{t+1} = \frac{1}{\beta}\mathsf{k}_t - \frac{1-\beta}{\beta\alpha}\mathsf{c}_t + \frac{(1-\alpha)(1-\beta)}{\beta\alpha}\mathsf{e}_t \tag{126}$$

$$\frac{1}{C_{t-1}} = \beta \mathrm{E}_{t-1}\left\{\exp\left[\log\left(\frac{1+\tilde{r}_t}{C_t}\right)\right]\right\}$$

$$= \beta \exp\left\{\mathrm{E}_{t-1}[\log(1+\tilde{r}_t) - \log(C_t)]\right.$$

$$\left. + \frac{1}{2}\mathrm{Var}\left[\log\left(\frac{1+\tilde{r}_t}{C_t}\right)\right]\right\}$$

$$\mathrm{E}_{t-1}\{\log C_t\} - \log C_{t-1}$$

$$= \log \beta + \frac{1}{2}\mathrm{Var}\left[\log\left(\frac{1+\tilde{r}_t}{C_t}\right)\right]$$

$$+ \mathrm{E}_{t-1}\log(1+\tilde{r}_t)$$

$$\mathrm{E}_{t-1}\{\log C_t - \log \bar{C}\} - (\log C_{t-1} - \log \bar{C})$$

$$= \mathrm{E}_{t-1}\{\log(1+\tilde{r}_t) - \log(1+\bar{r})\} + \chi_0$$

$$\mathrm{E}_{t-1}\mathsf{c}_t - \mathsf{c}_{t-1} = \mathrm{E}_{t-1}\tilde{\mathsf{r}}_t + \chi_0,$$

$$\mathrm{E}_{t-1}\mathsf{c}_t - \mathsf{c}_{t-1} = \mathrm{E}_{t-1}\tilde{\mathsf{r}}_t. \tag{127}$$

$$1 + \tilde{r}_t = 1 + \bar{r} + \alpha(\alpha - 1)\left(\frac{\bar{K}}{\bar{E}}\right)^{\alpha-1}\left(\frac{\mathrm{d}K_t}{\bar{K}} - \frac{\mathrm{d}E_t}{\bar{E}}\right),$$

$$\tilde{\mathsf{r}}_t = (1-\alpha)(1-\beta)\left(\mathsf{e}_t - \mathsf{k}_t\right). \tag{128}$$

$$\mathrm{E}_{t-1}\mathsf{c}_t - \mathsf{c}_{t-1} = (1-\alpha)(1-\beta)(\mathrm{E}_{t-1}\mathsf{e}_t - \mathsf{k}_t) \tag{129}$$

$$\mathsf{e}_t = \rho\mathsf{e}_{t-1} + \epsilon_t, \tag{130}$$

$$\mathsf{c}_t = a_{\mathsf{ck}}\mathsf{k}_t + a_{\mathsf{ce}}\mathsf{e}_t, \tag{131}$$

$$\mathsf{k}_{t+1} = \left[\frac{1}{\beta} - \frac{(1-\beta)a_{\mathsf{ck}}}{\beta\alpha}\right]\mathsf{k}_t + \left[\frac{(1-\alpha)(1-\beta)}{\beta\alpha} - \frac{(1-\beta)a_{\mathsf{ce}}}{\beta\alpha}\right]\mathsf{e}_t \tag{132}$$

$$a_{\mathsf{ck}}(\mathsf{k}_{t+1} - \mathsf{k}_t) + a_{\mathsf{ce}}(\mathrm{E}_t\mathsf{e}_{t+1} - \mathsf{e}_t) = (1-\alpha)(1-\beta)(\mathrm{E}_t\mathsf{e}_{t+1} - \mathsf{k}_{t+1}) \tag{133}$$

$$a_{\mathsf{ck}}\left[\frac{1-\beta}{\beta}-\frac{(1-\beta)a_{\mathsf{ck}}}{\beta\alpha}\right]\mathsf{k}_t$$

$$+\,a_{\mathsf{ck}}\left[\frac{(1-\alpha)(1-\beta)}{\beta\alpha}-\frac{(1-\beta)a_{\mathsf{ce}}}{\beta\alpha}\right]\mathsf{e}_t+a_{\mathsf{ce}}(\rho-1)\mathsf{e}_t$$

$$=\rho(1-\alpha)(1-\beta)\mathsf{e}_t-(1-\alpha)(1-\beta)\left[\frac{1}{\beta}-\frac{(1-\beta)a_{\mathsf{ck}}}{\beta\alpha}\right]\mathsf{k}_t$$

$$-\,(1-\alpha)(1-\beta)\left[\frac{(1-\alpha)(1-\beta)}{\beta\alpha}-\frac{(1-\beta)a_{\mathsf{ce}}}{\beta\alpha}\right]\mathsf{e}_t \qquad (134)$$

$$-\,a_{\mathsf{ck}}^2+[2\alpha-1+\beta(1-\alpha)]a_{\mathsf{ck}}$$

$$+\,\alpha(1-\alpha)=0 \qquad (135)$$

$$a_{\mathsf{ce}}=\frac{-a_{\mathsf{ck}}(1-\alpha)+(1-\alpha)\big[\rho\beta\alpha-(1-\alpha)(1-\beta)\big]}{\frac{\beta\alpha}{1-\beta}(\rho-1)-\big[a_{\mathsf{ck}}+(1-\alpha)(1-\beta)\big]}$$

$$1 = \beta \mathrm{E}_t \left\{ \left[1 + \alpha \left(\frac{E_{t+1}}{K_{t+1}} \right)^{1-\alpha} \right] \left(\frac{C_t}{C_{t+1}} \right) \right\},$$

$$1 = \beta \mathrm{E}_t \left\{ \left[1 + \alpha \left(\frac{E^*_{t+1}}{K^*_{t+1}} \right)^{1-\alpha} \right] \left(\frac{C_t}{C_{t+1}} \right) \right\}.$$

$$\mathrm{E}_{t-1}\mathsf{c}_t - \mathsf{c}_{t-1} = (1-\alpha)(1-\beta)(\mathrm{E}_{t-1}\mathsf{e}_t - \mathsf{k}_t),$$

$$\mathrm{E}_{t-1}\mathsf{c}_t - \mathsf{c}_{t-1} = (1-\alpha)(1-\beta)(\mathrm{E}_{t-1}\mathsf{e}^*_t - \mathsf{k}^*_t)$$

$$\mathrm{E}_{t-1}(\mathsf{e}_t - \mathsf{k}_t) = \mathrm{E}_{t-1}(\mathsf{e}_t^* - \mathsf{k}_t^*). \tag{136}$$

$$K_{t+h} - K_t = shF(K_t, E_t L_t) - \delta h K_t, \tag{3'}$$

$$E_{t+h} - E_t = ghE_t, \tag{4'}$$

$$L_{t+h} - L_t = nhL_t. \tag{5'}$$

$$\frac{k^{\mathrm{E}}_{t+h} - k^{\mathrm{E}}_{t}}{h} = \frac{sf(k^{\mathrm{E}}_{t})}{(1+nh)(1+gh)} - \frac{(n+g+\delta)+ngh}{(1+nh)(1+gh)}k^{\mathrm{E}}_{t}$$

$$sf(\bar{k}^{\mathrm{E}}) = [(n+g+\delta)+ngh]\bar{k}^{\mathrm{E}}.$$

$$sf(\bar{k}^{\mathrm{E}}) = (n+g+\delta)\bar{k}^{\mathrm{E}}, \tag{9'}$$

$$\dot{K}_t = sF(K_t, E_t L_t) - \delta K_t, \tag{3''}$$

$$\frac{\dot{E}_t}{E_t} = g, \tag{4''}$$

$$\frac{\dot{L}_t}{L_t} = n, \tag{5''}$$

$$\dot{X}_t \equiv \frac{\mathrm{d}X_t}{\mathrm{d}t} = \lim_{h\to 0} \frac{X_{t+h} - X_t}{h}.$$

$$\frac{\dot{K}_t}{E_t L_t} = \frac{sF(K_t, E_t L_t)}{E_t L_t} - \frac{\delta K_t}{E_t L_t}.$$

$$\dot{k}_t^{\mathrm{E}} = \frac{\dot{K}_t}{E_t L_t} - \frac{K_t}{E_t L_t}\left(\frac{\dot{E}_t}{E_t} + \frac{\dot{L}_t}{L_t}\right) = \frac{\dot{K}_t}{E_t L_t} - k_t^{\mathrm{E}}(g + n),$$

$$\dot{k}_t^{\mathrm{E}} = sf(k_t^{\mathrm{E}}) - (n + g + \delta)k_t^{\mathrm{E}}.$$

$$E_t = (1 + gh)^{t/h} E_0,$$

$$\lim_{h\to 0} E_t = \lim_{h\to 0}(1+gh)^{t/h}E_0$$

$$= \lim_{n\to\infty}(1+\frac{g}{n})^{tn}E_0 = \exp\,(gt)\;E_0$$

$$U_t = \sum_{s=t}^{\infty}\left(\frac{1}{1+\delta h}\right)^{(s-t)/h} u(C_s)h,$$

$$K_{t+h} = K_t + hF(K_t) - hC_t$$

$$\max_{\{K_s\}}\sum_{s=t}^{\infty}\left(\frac{1}{1+\delta h}\right)^{(s-t)/h} u\left[\frac{K_s - K_{s+h}}{h} + F(K_s)\right]h,\; K_t \text{ given}$$

$$u'(C_s) = \left(\frac{1}{1+\delta h}\right)\left[1 + hF'(K_{s+h})\right]u'(C_{s+h})$$

$$\frac{u'(C_{s+h}) - u'(C_s)}{h} = \left[\frac{\delta}{1+\delta h} - \frac{F'(K_{s+h})}{1+\delta h}\right]u'(C_{s+h})$$

$$\frac{\mathrm{d}u'(C_s)}{\mathrm{d}C_s} \cdot \frac{\mathrm{d}C_s}{\mathrm{d}s} = u''(C_s)\dot{C}_s = \left[\delta - F'(K_s)\right]u'(C_s).$$

$$U_t = \int_t^{\infty} u(C_s)\exp[-\delta(s-t)]\mathrm{d}s$$

$$\dot{K}_s = F(K_s) - C_s.$$

$$\mathcal{H}(C_s, K_s, s) = u(C_s) + \lambda_s \left[F(K_s) - C_s \right],$$

$$\frac{\partial \mathcal{H}}{\partial C_s} = u'(C_s) - \lambda_s = 0, \quad \dot{\lambda}_s = \delta \lambda_s - \frac{\partial \mathcal{H}}{\partial K_s}$$

$$= \lambda_s[\delta - F'(K_s)]$$

$$\dot{\lambda}_s = u''(C_s)\dot{C}_s = [\delta - F'(K_s)]u'(C_s),$$

$$Y_t = A_t K_t^{\alpha} L_t{}^{1-\alpha}, \tag{137}$$

$$U_t = \log(c_t^{\mathrm{Y}}) + \beta \mathrm{E}_t \log(c_{t+1}^{\mathrm{O}}), \tag{138}$$

$$\begin{aligned} c_{t+1}^{\mathrm{O}} = (w_t - c_t^{\mathrm{Y}})[x_{t+1}(1 + r_{t+1}) \\ + (1 - x_{t+1})(1 + \tilde{r}_{t+1})] \end{aligned} \tag{139}$$

$$c_t^{\mathrm{Y}} = \frac{w_t}{1+\beta}.$$

$$s_t^{\mathrm{Y}} = \frac{\beta w_t}{1+\beta}. \tag{140}$$

$$K_{t+1} = L_t s_t^{\mathrm{Y}}.$$

$$k_{t+1} = \frac{\beta w_t}{(1+\beta)(1+n)}.$$

$$k_{t+1} = \frac{\beta(1-\alpha)A_t k_t^{\alpha}}{(1+\beta)(1+n)} \tag{141}$$

$$\mathsf{k}_{t+1} = \log\left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right] + \alpha\mathsf{k}_t + \mathsf{a}_t \quad (142)$$

$$\mathsf{k}_t = \frac{\mathsf{y}_t - \mathsf{a}_t}{\alpha}.$$

$$\mathsf{y}_t = \chi_0 + \alpha\mathsf{y}_{t-1} + \mathsf{a}_t, \quad (143)$$

$$\chi_0 \equiv \alpha \log \frac{\beta(1-\alpha)}{(1+\beta)(1+n)}.$$

Table 7.1 Convergence in Output Per Capita, 1870–1979

Country	Per Capita 1870 Income (1975 dollars)	Growth in Per Capita Income (1870–1979, log difference $\times 100$)
Australia	1,922	116
United Kingdom	1,214	145
Switzerland	1,118	174
Belgium	1,137	168
Netherlands	1,104	166
United States	1,038	207
Denmark	883	201
Canada	881	214
France	847	207
Austria	751	203
Italy	746	178
West Germany	731	223
Norway	665	228
Sweden	557	247
Finland	506	241
Japan	328	286

Sources: De Long (1988) and Maddison (1982).

Table 7.2 The Once Rich Seven

Country	**Per Capita 1870 Income (1975 dollars)**	**Growth in Per Capita Income (1870–1979, log difference** $\times$ **100)**
New Zealand	981	157
Argentina	762	141
East Germany	741	199
Spain	728	176
Ireland	656	167
Portugal	637	150
Denmark	519	150

Source: De Long (1988).

Table 7.3 Average Annual Total Factor Productivity Growth in East Asia and the G-7 Countries

Country	**Period**	**Annual Growth (percent)**
Hong Kong	1966–91	2.3
Singapore	1966–90	0.2
South Korea	1966–90	1.7
Taiwan	1966–90	2.1
Canada	1960–89	0.5
France	1960–89	1.5
Germany	1960–89	1.6
Italy	1960–89	2.0
Japan	1960–89	2.0
United Kingdom	1960–89	1.3
United States	1960–89	0.4

Source: Young (1995).

Table 7.4 World Population Growth, 1,000,000 B.C. to 1990

Start of Period	Population (millions)	Population Growth Rate (percent per year)	Major Calamities
−1,000,000	0.125	0.0003	
−25,000	3.34	0.0020	
−5000	5	0.0562	
−2000	27	0.0873	
−500	100	0.1062	
1	170	0.0559	
200	190	0.0	
400	190	0.0256	
600	200	0.0477	
800	220	0.0931	
1000	265	0.1533	
1200	360	0.0	Mongol invasions
1300	360	−0.0282	Black Death
1400	350	0.2217	
1600	545	0.1127	Thirty Years War, Ming dynasty fall
1700	610	0.3897	
1800	900	0.5926	
1900	1625	1.0125	
1980	4450	1.8101	