Foundations of International Macroeconomics[1]

Workbook[2]

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Chapter 7 Solutions

1. (The word "equiproportionate" in the third line of the statement of this exercise should be "lump-sum.")

(a) With the introduction of tax-financed government spending in the Weil (1989a) model, the period budget constraint for a family of vintage v is given by

$$k_{t+1}^{v} = (1 + r_t)\, k_t^{v} + w_t - \tau - c_t^{v}$$

(where τ is the lump-sum tax) instead of by eq. (30) in Chapter 7. We write the above expression in average per capita terms as

$$k_{t+1} - k_t = \frac{f(k_t) - c_t - g}{1+n} - \frac{nk_t}{1+n}, \qquad (1)$$

giving the analog of eq. (32) in the chapter. Here we have used the balanced-budget constraint $g = \tau$. The introduction of government expenditure therefore shifts the $\Delta k = 0$ locus down by $g/(1+n)$ in the phase diagram for per capita consumption and the capital-labor ratio (figure 7.7 on p. 449). The presence of tax-financed government spending does not affect eq. (33) in the text:

$$c_{t+1} = [1 + f'(k_{t+1})]\, [\beta c_t - n(1-\beta) k_{t+1}]\,. \qquad (2)$$

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[2] Version 1.1, February 27, 1998. For online updates and corrections, see http://www.princeton.edu/ObstfeldRogoffBook.html

(b) Figure 7.1 shows the phase diagram. In that figure, an unanticipated permanent rise in g shifts the $\Delta k = 0$ schedule down. At the instant the shock occurs, consumption declines immediately from point **A** to point **B**. In subsequent periods, there is a gradual decumulation of capital, and consumption continues to decline until the new steady state **A′** is reached at lower levels of $\bar{c}$ and $\bar{k}$. The decline in $\bar{k}$ represents a "crowding out" effect of balanced-budget government spending.

(c) As can be seen in figure 7.2, the impact effect of the announcement is an immediate decline in consumption from c_0 to c_1. The economy then moves along an "unstable" path relative to the initial steady state. Along that path, c gradually declines while k increases until the economy reaches point **D** on date T. Point **D** lies on the stable path corresponding to the new constant level of g. After date T, c and k both decline until the new steady-state $\left(\bar{c}_{\text{NEW}}, \bar{k}_{\text{NEW}}\right)$ is reached (point **A′**). Per capita consumption and the capital-labor ratio are lower in the new steady state relative to the steady state with $g = 0$.

2. (Note that there is an obvious typo in the question. The production function is Ak^{α} and not Ak.) The economy's initial equilibrium is the autarky steady state as in the discussion in section 7.2.2.3 in the book. The exercise states that in that steady state the autarky interest rate r^{A} equals the world interest rate r. (In particular, this means that initially the economy is not borrowing-constrained.) For an arbitrary productivity parameter A in the production function, the autarky steady-state capital stock is given by

$$\bar{k}^{\text{A}} = \frac{\beta \bar{w}^{\text{A}}}{1+\beta} = \frac{\beta(1-\alpha)A\left(k^{\text{A}}\right)^{\alpha}}{1+\beta}$$

or

$$\bar{k}^{\text{A}} = \left[\frac{A\beta(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}}.$$

Using the last expression to substitute for $\bar{k}^{\text{A}}$, we see that the equality of the world and steady-state autarky interest rates is written as

$$\begin{aligned} \bar{r}^{\mathrm{A}} &= \alpha A \left(\bar{k}^{\mathrm{A}}\right)^{\alpha-1} - 1 \\ &= \frac{\alpha(1+\beta)}{(1-\alpha)\beta} - 1 \\ &= r, \end{aligned} \tag{3}$$

where we recall that the rate of depreciation, δ, is 100 percent. Notice that the second and third equalities above are *independent* of the parameter A. This means that after A rises permanently from $A = 1$, the economy necessarily returns to a new unconstrained steady state with $r^{\mathrm{A}} = r^{\mathrm{D}} = r$, but with a higher capital stock. Another way to see that result is to note that the equality

$$\frac{\beta \bar{w}}{1+\beta} \geq \bar{k}^{\mathrm{U}}$$

ensures convergence to the world interest rate [cf. eq. (60) in Chapter 7; $\bar{w}$ is the unconstrained steady-state wage]. But it us easy to check that a rise in A from $A = 1$ simply multiplies both $\bar{w}$ and $\bar{k}^{\mathrm{U}}$ above by the same factor $A^{1/(1-\alpha)}$, leaving the preceding inequality intact if it also held before the rise in productivity.

The economy can move to its new steady state in a single period only if

$$\frac{\beta w_0}{1+\beta} + \eta w_0 \geq \left(\frac{A\alpha}{1+r}\right)^{\frac{1}{1-\alpha}}, \tag{4}$$

where the right-hand side is the new steady-state capital stock and w_0 is the wage in the first period A rises, given the predetermined capital stock:

$$w_0 = (1-\alpha)A\left[\frac{\beta(1-\alpha)}{1+\beta}\right]^{\frac{\alpha}{1-\alpha}}.$$

Using eq. (3), we may express inequality (4) as

$$\left(\frac{\beta}{1+\beta} + \eta\right)(1-\alpha)A\left[\frac{\beta(1-\alpha)}{1+\beta}\right]^{\frac{\alpha}{1-\alpha}} \geq \left[\frac{A\beta(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}},$$

which is equivalent to

$$\left(\frac{\beta}{1+\beta}+\eta\right) \geq A^{\frac{\alpha}{1-\alpha}}\left(\frac{\beta}{1+\beta}\right).$$

This inequality is quite intuitive. If the productivity increase is very small, the economy will reach its new (unconstrained) steady state in a single period with even a small amount of loans from abroad (η small). If, in contrast, A rises by a very large amount, convergence will be slower, because in the short run wages rise only by a factor of A, whereas the long-run unconstrained capital stock rises by the bigger factor $A^{1/(1-\alpha)}$. (The ratio of the two is the term $A^{\alpha/(1-\alpha)}$ that appears in the preceding inequality.)

3. The planner's problem is to maximize

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \log C_s \tag{5}$$

subject to the production technology in the R&D sector,

$$A_{t+1} - A_t = \theta A_t L_{A,t}, \tag{6}$$

labor-market clearing,

$$L = L_A + L_Y, \tag{7}$$

the final-goods production function,

$$Y_t = L_{Y,t}^{1-\alpha} A_t K_t^{\alpha}, \tag{8}$$

and the social budget constraint

$$Y_t = C_t + A_{t+1} K_{t+1}. \tag{9}$$

The Lagrangian for the maximization problem is

$$\mathcal{L} = \sum_{s=t}^{\infty} \beta^{s-t} \Big\{ \log[A_s K_s^{\alpha} (L - L_{A,s})^{1-\alpha} - A_{s+1} K_{s+1}] + \lambda_s (A_{s+1} - A_s - \theta A_s L_{A,s}) \Big\}$$

(see footnote 42 on p. 491). The previous expression follows from using (6), (7), (8), and (9) to substitute for C in (5). Next we take derivatives with respect to $L_{A,t}, K_{t+1}$ and A_{t+1}:

$$L_{A,t}: \quad \left[\frac{A_t K_t^{\alpha}(\alpha-1)(L-L_{A,t})^{-\alpha}}{C_t}\right] = \theta\lambda_t A_t,$$

$$A_{t+1}: \quad \left(\frac{-K_{t+1}}{C_t}\right) + \lambda_t = \beta\left\{\lambda_{t+1}\left(1+\theta L_{A,t+1}\right) - \left[\frac{K_{t+1}^{\alpha}(L-L_{A,t+1})^{1-\alpha}}{C_{t+1}}\right]\right\},$$

$$K_{t+1}: \quad \left(\frac{A_{t+1}}{C_t}\right) = \frac{\beta\left[A_{t+1}(L-L_{A,t+1})^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\right]}{C_{t+1}}.$$

We are looking for a steady-state equilibrium with constant real interest rate, constant K, constant relative prices, and a constant allocation of labor across the two sectors; that is, we are looking for a balanced growth path. Output and consumption will grow at a rate g in steady state. Along a steady-state path the preceding first-order condition with respect to K_{t+1} can be written more simply as:

$$1+g = \beta\left[(L-L_A)^{1-\alpha}\,\alpha K^{\alpha-1}\right]. \tag{10}$$

The first-order condition with respect to $L_{A,t}$ simplifies to:

$$\lambda_t = \frac{(\alpha-1)K^{\alpha}\left(L-L_A\right)^{-\alpha}}{\theta C_t}. \tag{11}$$

Substitute (11) into the first-order condition for A_{t+1} (with $K_{t+1} = K$ in steady state). Then multiply through by C_{t+1} to obtain

$$\begin{aligned} & -(1+g)K + \frac{(1+g)(\alpha-1)K^{\alpha}(L-L_A)^{-\alpha}}{\theta} \qquad (12) \\ = \; & \frac{\beta\,(\alpha-1)\,(1+\theta L_A)K^{\alpha}\left(L-L_A\right)^{-\alpha}}{\theta} - \beta\left[K^{\alpha}(L-L_A)^{1-\alpha}\right]. \end{aligned}$$

Also,

$$\frac{A_{t+1}-A_t}{A_t} = \theta L_A = g. \tag{13}$$

[Equation (13) follows from (6).] Along a balanced growth path, the number of capital good types grows at rate g, whereas the quantity of each type of capital good remains constant at K. Finally, using (10), (12), and (13), we can solve for g, K, and L_A:

$$g = \beta\theta L - (1-\beta),$$

$$K = \alpha^{\frac{1}{1-\alpha}} \left(\frac{1-\beta}{\theta}\right) (1+\theta L)^{\frac{\alpha}{\alpha-1}},$$

$$L_A = L - \frac{(1+g)(1-\beta)}{\theta\beta}.$$

[Hint: To solve for g, divide (12) through by $K^{\alpha}(L-L_A)^{1-\alpha}$, then use (10) and (13).] As expected, the growth rate for the planner's problem is unambiguously higher than in the laissez-faire equilibrium. This discrepancy arises because in the decentralized solution there are two distortions. First, firms in the R&D sector do not internalize the fact that their inventions will lower the cost of producing future inventions. Second, for any given allocation of labor, monopolistic suppliers set K lower than the planner would. That is, they underutilize inventions, creating a static inefficiency (albeit one that affects R&D employment).

4. In section 7.4.1, we saw that for logarithmic utility equilibrium consumption is

$$C_t = (1-\alpha\beta)A_t K_t^{\alpha}.$$

We simply check that the consumption function in the question also holds true. Note that $w_s = (1-\alpha)\, A_s K_s^{\alpha}$ and $\tilde{r}_t = \alpha A_t K_t^{\alpha-1} - 1$ (due to the assumption of 100 percent capital depreciation in the first period of use). Also valid under the preceding solution for consumption is the equality

$$\frac{u'(C_s)}{u'(C_t)} = \frac{C_t}{C_s} = \frac{A_t K_t^{\alpha}}{A_s K_s^{\alpha}}$$

for all t and s, implying that

$$\begin{aligned}
\sum_{s=t}^{\infty} \mathrm{E}_t \left\{ \frac{\beta^{s-t} u'(C_s)}{u'(C_t)} w_s L \right\} &= (1-\alpha) A_t K_t^{\alpha} \sum_{s=t}^{\infty} \beta^{s-t} \\
&= \frac{1-\alpha}{1-\beta} A_t K_t^{\alpha}.
\end{aligned}$$

(Recall that L was normalized to equal 1 in section 7.4.1.) Simple substitution therefore yields the required result,

$$C_t = (1-\beta) \left(\alpha A_t K_t^{\alpha} + \frac{1-\alpha}{1-\beta} A_t K_t^{\alpha} \right) = (1-\alpha\beta) A_t K_t^{\alpha}.$$

Interpretation*:* Under log utility optimal consumption is a fraction $(1-\beta)$ of lifetime wealth; see supplement A to Chapter 5. The latter, in turn, is the sum of nonhuman wealth $(1+\tilde{r}_t)K_t$ and human wealth

$$\sum_{s=t}^{\infty} \mathrm{E}_t \left\{ \frac{\beta^{s-t} u'(C_s)}{u'(C_t)} w_s L \right\}$$

(the present discounted value of current and future labor earnings).