Foundations of International Macroeconomics[1]

Workbook[2]

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Chapter 5 Solutions

1. Since consumption on date 2 is the sum of endowment and payments of contingent assets for the realized state, we have

$$B_2(s) = C_2(s) - Y_2(s), \;\; s = 1, 2.$$

For $s = 1$, premultiply by $p(1)/(1+r)$ and use eq. (16) in Chapter 5 to obtain the following:

$$\begin{aligned}
\frac{p(1)}{(1+r)}B_2(1) &= \frac{\pi(1)\beta}{1+\beta}\left[Y_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right] - \frac{p(1)}{1+r}Y_2(1) \\
&= \pi(1)\frac{\beta}{1+\beta}Y_1 - \pi(1)\frac{1}{1+\beta}\left[\frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right] \\
&\quad + \pi(1)\left[\frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right] - \frac{p(1)}{1+r}Y_2(1) \\
&= \pi(1)\frac{\beta}{1+\beta}Y_1 - \pi(1)\frac{1}{1+\beta}\left[\frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right] \\
&\quad + \pi(1)\frac{p(2)Y_2(2)}{1+r} - \pi(2)\frac{p(1)Y_2(1)}{1+r} \\
&= \pi(1)CA_1 + \frac{p(2)\pi(2)Y_2(1)}{1+r}\left[\frac{\pi(1)/Y_2(1)}{\pi(2)/Y_2(2)} - \frac{p(1)}{p(2)}\right].
\end{aligned}$$

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Here, the last equality comes from eq. (17), Chapter 5. Using eq. (80) from the chapter, we see that the autarky price of the Arrow-Debreu security for state 1 relative to that of the state 2 security is

$$\frac{p(1)^{\mathrm{A}}}{p(2)^{\mathrm{A}}} = \frac{\pi(1)/Y_2(1)}{\pi(2)/Y_2(2)}.$$

Substitution of the preceding into the expression for $p(1)B_2(1)/(1+r)$, gives the required result. The result for $B_2(2)$ follows from the identity

$$CA_1 = \frac{p(1)}{1+r}B_2(1) + \frac{p(2)}{1+r}B_2(2).$$

The statement of the exercise provides the intuition.

2. The necessary first-order conditions are

$$\frac{p(s)}{1+r}u^{'}(C_1) = \pi(s)\beta u^{'}[C_2(s)], \;\; s = 1, 2.$$

For our utility function, $u^{'}(C_1) = 1/C_1$ and $u^{'}[C_2(s)] = 1$, so that the above conditions imply

$$\frac{p(s)}{1+r} = \pi(s)\beta C_1, \;\; s = 1, 2. \tag{1}$$

(A similar relation holds for C_1^*, the initial consumption of Foreign residents.) If we divide eq. (1) for $s = 1$ by its analog for $s = 2$, we see that

$$\frac{\pi(1)}{\pi(2)} = \frac{p(1)}{p(2)},$$

implying that equilibrium prices must be actuarially fair. Since $p(1)+p(2) = 1$, it follows that the Arrow-Debreu prices equal the respective probabilities of the state occurring:

$$p(s) = \pi(s), \;\; s = 1, 2.$$

Assuming that Home and Foreign share the same discount factor β, we may add eq. (1) for Home and for Foreign to obtain

$$C_1 + C_1^* = Y_1^{\mathrm{W}} = \frac{2p(1)}{(1+r)\pi(1)\beta}.$$

Because $p(1) = \pi(1)$, we obtain an expression for the world interest rate

$$1 + r = \frac{2}{\beta Y_1^{\mathrm{W}}}.$$

Using Euler eq. (1) again, but substituting in this expression for $1 + r$, we see that equilibrium date 1 consumptions are:

$$C_1 = C_1^* = \frac{Y_1^{\mathrm{W}}}{2}.$$

The equilibrium intertemporal budget constraint of a Home resident is

$$C_1 + \frac{\pi(1)}{1+r}C_2(1) + \frac{\pi(2)}{1+r}C_2(2) = Y_1 + \frac{\pi(1)}{1+r}Y_2(1) + \frac{\pi(2)}{1+r}Y_2(2),$$

so that date 2 consumptions must obey

$$\frac{\pi(1)}{1+r}C_2(1) + \frac{\pi(2)}{1+r}C_2(2) = Y_1 - \frac{Y_1^{\mathrm{W}}}{2} + \frac{\pi(1)}{1+r}Y_2(1) + \frac{\pi(2)}{1+r}Y_2(2); \qquad (2)$$

there is a similar equation for Foreign. Since utility is linear in date 2 consumption with weights $\pi(1)$ and $\pi(2)$, a Home resident is indifferent between any pair $[C_2(1), C_2(2)]$ satisfying eq. (2). On date 2, however, goods-market equilibrium requires that

$$C_2(s) + C_2^*(s) = Y_2^{\mathrm{W}}(s), \;\; s = 1, 2.$$

Thus, we have four equations—eq. (2) and its Foreign analog, plus the state 1 and 2 equilibrium conditions—to determine the four unknowns $[C_2(s), C_2^*(s)]$, $s = 1, 2$. If you play with these four equations, however, you will realize that the date 2 consumption allocation actually is *indeterminate.* To see this intuitively, imagine any equilibrium allocation of date consumption across states. Suppose now that Home were to reduce its state 1 consumption by $1/\pi(1)$ units and raise its state 2 consumption by $1/\pi(2)$ units. Constraint (2) would still be satisfied and Home residents would feel no worse off. If Foreign were simultaneously to raise its state 1 consumption by $1/\pi(1)$ units while lowering its state 2 consumption by $1/\pi(2)$ units—an action which is feasible for Foreign and does not lower its welfare—date 2 contingent commodity markets would still clear. Thus, the date 2 consumption allocation is not fully determined in the model. This indeterminacy result is, of course, a consequence of risk-neutrality with regard to date 2. People care about the expected values of second-period payoffs, but not about their distribution across states. This indifference explains why the national allocations of second-period consumption across the states of nature is not tied down.

3. (a) Ignoring nonnegativity, write the unconstrained maximization as

$$\max_{B_2}\ [(1+r)B_1 - B_2 + Y_1] - \frac{a_0}{2}\left[(1+r)B_1 - B_2 + Y_1\right]^2$$
$$+\frac{1}{1+r}\mathrm{E}_1\left\{[(1+r)B_2 + Y_2(s)] - \frac{a_0}{2}\left[(1+r)B_2 + Y_2(s)\right]^2\right\}.$$

The first-order condition for B_2 is:

$$C_1 = \mathrm{E}_1\left\{C_2(s)\right\}.$$

The $\mathcal{S}+1$ budget constraints in the problem imply that

$$\mathrm{E}_1\left\{C_1 + \frac{C_2(s)}{1+r}\right\} = \mathrm{E}_1\left\{(1+r)B_1 + Y_1 + \frac{Y_2(s)}{1+r}\right\}.$$

Thus by substitution of the first-order condition,

$$\left(1+\frac{1}{1+r}\right)C_1 = \mathrm{E}_1\left\{(1+r)B_1 + Y_1 + \frac{Y_2(s)}{1+r}\right\},$$

that is,

$$C_1 = \frac{1+r}{2+r}\mathrm{E}_1\left\{(1+r)B_1 + Y_1 + \frac{Y_2(s)}{1+r}\right\}. \tag{3}$$

For the ∞-horizon case, we just get the usual "permanent-income" formula, essentially eq. (32) of Chapter 2 (suitably adapted).

(b) The consumption formula of part a will be generally valid if the nonnegativity constraint on consumption never binds, that is, if, even when output hits its minimal date 2 value (in state $s = 1$), $C_2 \geq 0$. This last inequality will hold if and only if

$$(1+r)B_2 + Y_2(1) \geq 0$$

for $B_2 = (1+r)B_1 + Y_1 - C_1$, where C_1 is given by eq. (3) above. That is, we must have

$$(1+r)\left\{(1+r)B_1 + Y_1 - \frac{1+r}{2+r}\left[(1+r)B_1 + Y_1 + \frac{\mathrm{E}_1 Y_2}{1+r}\right]\right\} + Y_2(1) \geq 0$$

which is equivalent to

$$(1+r)B_1 + Y_1 + \frac{2+r}{1+r}Y_2(1) \geq \mathrm{E}_1 Y_2.$$

If this inequality does not hold, then the nonnegativity constraint on C_2 binds in at least one state of nature on date 2, so we cannot ignore the associated Kuhn-Tucker multiplier (see supplement A to Chapter 2). In that case, the Kuhn-Tucker theorem predicts that date 1 consumption must make $C_2(1) = 0$ (in state 1 of date 2 when output is minimal). Since

$$C_2(1) = (1+r)\left[(1+r)B_1 + Y_1 - C_1\right] + Y_2(1) = 0$$

therefore holds, we see that $C_1 = (1+r)B_1 + Y_1 + Y_2(1)/(1+r)$.

(c) The state-by-state Euler equations are

$$\frac{p(s)}{1+r}(1 - a_0 C_1) = \frac{\pi(s)}{1+r}\left[1 - a_0 C_2(s)\right],$$

which reduce to

$$C_1 = C_2(s),\ \forall s,$$

because we've assumed $p(s) = \pi(s)$. Thus consumption is constant across states and dates, equal to $\bar{C}$, given by

$$\bar{C} = \left(\frac{1+r}{2+r}\right)\left[Y_1 + \sum_{s=1}^{\mathcal{S}} \frac{p(s)Y_2(s)}{1+r}\right] = \left(\frac{1+r}{2+r}\right)\left[Y_1 + \sum_{s=1}^{\mathcal{S}} \frac{\pi(s)Y_2(s)}{1+r}\right].$$

The critical difference between the equation above and eq. (3) is that the preceding equation holds ex post as well as ex ante, i.e., it holds in every state on date 2 as well as on date 1. Equation (3) above, in contrast, implies that date 2 consumption varies one-for-one with the output realization (date 2 consumption is not insured in the "bonds-only" asset regime). Thus the possibility of negative consumption is an issue in the bonds-only case, though not under complete markets.

4. Output-market equilibrium on date 1 requires that

$$\sum_{n=1}^{N} C_1^n = \frac{1}{1+\beta}\sum_{n=1}^{N}(Y_1^n + V_1^n) = \sum_{n=1}^{N} Y_1^n,$$

or

$$\sum_{n=1}^{N} V_1^n = \beta \sum_{n=1}^{N} Y_1^n = \beta Y_1^{\mathrm{W}}. \tag{4}$$

(It is straightforward to check that if the preceding condition holds, the output market also clears on date 2, in every state s.) Next we have to find equilibrium asset prices under condition (4), and check that they are indeed consistent with (4). Under (4) and the conjectured solutions for consumption, an agent in any country n has a marginal rate of substitution between date 1 consumption and date 2, state s consumption, of

$$\frac{\beta u'\left[C_2^n(s)\right]}{u'(C_1^n)} = \frac{\beta C_1^n}{C_2^n(s)} = \frac{\sum_{m=1}^{N} V_1^m}{\sum_{m=1}^{N} Y_2^m(s)} = \frac{\beta Y_1^{\mathrm{W}}}{Y_2^{\mathrm{W}}(s)}.$$

This means that agents from any country n will be content to hold the available country mutual funds at prices

$$V_1^m = \sum_{s=1}^{\mathcal{S}} \pi(s)\frac{\beta u'\left[C_2^n(s)\right]}{u'(C_1^n)} Y_2^m(s) = \sum_{s=1}^{\mathcal{S}} \pi(s)\left[\frac{\beta Y_1^{\mathrm{W}}}{Y_2^{\mathrm{W}}(s)}\right] Y_2^m(s),\ m = 1,\ \ldots,\ N.$$

At these prices,

$$\sum_{m=1}^{N} V_1^m \quad = \quad \sum_{m=1}^{N}\left\{\sum_{s=1}^{\mathcal{S}} \pi(s)\left[\frac{\beta Y_1^{\mathrm{W}}}{Y_2^{\mathrm{W}}(s)}\right] Y_2^m(s)\right\}$$

$$
\begin{aligned}
&= \sum_{s=1}^{\mathcal{S}} \left\{ \pi(s) \left[\frac{\beta Y_1^{\mathrm{W}}}{Y_2^{\mathrm{W}}(s)} \right] \left[\sum_{m=1}^{N} Y_2^m(s) \right] \right\} \\
&= \beta Y_1^{\mathrm{W}},
\end{aligned}
$$

so indeed, condition (4) is satisfied. It remains only to calculate the riskless rate of interest. That comes from the Euler equation,

$$
\frac{1}{1+r} = \sum_{s=1}^{\mathcal{S}} \pi(s) \frac{\beta u' \left[C_2^n(s) \right]}{u'(C_1^n)} = \sum_{s=1}^{\mathcal{S}} \pi(s) \frac{\beta C_1^n}{C_2^n(s)}.
$$

But the consumption functions, together with (4), imply that for any country n,

$$
\begin{aligned}
C_1^n &= \frac{1}{1+\beta} \left(\frac{Y_1^n + V_1^n}{Y_1^{\mathrm{W}} + \sum_{m=1}^{N} V_1^m} \right) \left(Y_1^{\mathrm{W}} + \sum_{m=1}^{N} V_1^m \right) \\
&= \left(\frac{Y_1^n + V_1^n}{Y_1^{\mathrm{W}} + \sum_{m=1}^{N} V_1^m} \right) Y_1^{\mathrm{W}} = \mu^n Y_1^{\mathrm{W}},
\end{aligned}
$$

$$
C_2^n(s) = \left(\frac{Y_1^n + V_1^n}{Y_1^{\mathrm{W}} + \sum_{m=1}^{N} V_1^m} \right) Y_2^{\mathrm{W}}(s) = \mu^n Y_2^{\mathrm{W}}(s),
$$

allowing us to express the equilibrium real interest rate in terms of exogenous variables through

$$
\frac{1}{1+r} = \sum_{s=1}^{\mathcal{S}} \pi(s) \frac{\beta Y_1^{\mathrm{W}}}{Y_2^{\mathrm{W}}(s)}.
$$

5. (a) With exponential utility the individual Euler equation for state s is

$$
\exp(-\gamma C_1) = \frac{(1+r)\beta\pi(s)}{p(s)} \exp\left[-\gamma C_2(s) \right],
$$

or, taking logs,

$$
C_1 = C_2(s) - \frac{1}{\gamma} \log \left[\frac{(1+r)\beta\pi(s)}{p(s)} \right].
$$

Summing over the two countries implies

$$
Y_1^{\mathrm{W}} = Y_2^{\mathrm{W}}(s) - \frac{2}{\gamma} \log \left[\frac{(1+r)\beta\pi(s)}{p(s)} \right],
$$

which can be solved for

$$\frac{p(s)}{1+r} = \beta\pi(s)\exp\left\{-\frac{\gamma}{2}\left[Y_2^{\mathrm{w}}(s) - Y_1^{\mathrm{w}}\right]\right\}. \tag{5}$$

Summing over states yields the (gross) interest rate,

$$1+r = \frac{\exp\left(-\frac{\gamma}{2}Y_1^{\mathrm{w}}\right)}{\beta\sum_s \pi(s)\exp\left[-\frac{\gamma}{2}Y_2^{\mathrm{w}}(s)\right]},$$

from which $p(s)$ is easily calculated.

(b) Notice that under the proposed consumption allocation markets clear on each date/state and, given the Arrow-Debreu prices calculated above, the complete-markets intertemporal Euler equations hold. For example,

$$\begin{aligned}
\exp(-\gamma C_1) &= \exp\left(-\frac{\gamma}{2}Y_1^{\mathrm{w}}\right)\exp(\gamma\mu) \\
&= \exp\left\{\frac{\gamma}{2}\left[Y_2^{\mathrm{w}}(s) - Y_1^{\mathrm{w}}\right]\right\}\exp\left[-\frac{\gamma}{2}Y_2^{\mathrm{w}}(s)\right]\exp(\gamma\mu) \\
&= \frac{(1+r)\beta\pi(s)}{p(s)}\exp\left[-\gamma C_2(s)\right],
\end{aligned}$$

where we have used (5) above. This shows efficiency. It is also easy to check (it is a special case of part c below) that the Euler equations for equity shares hold at the implied equilibrium values of V_1 and V_1^* (which also are given in part c). Now we check that the allocation satisfies budget constraints. Home's budget under the proposed equilibrium are

$$Y_1 + V_1 = \frac{1}{2}V_1 + \frac{1}{2}V_1^* + B_2 + C_1, \quad C_2(s) = \frac{1}{2}Y_2(s) + \frac{1}{2}Y_2^*(s) + (1+r)B_2,$$

whereas Foreign's are

$$Y_1^* + V_1^* = \frac{1}{2}V_1 + \frac{1}{2}V_1^* - B_2 + C_1^*, \quad C_2^*(s) = \frac{1}{2}Y_2(s) + \frac{1}{2}Y_2^*(s) - (1+r)B_2.$$

If we substitute

$$C_1 = \frac{1}{2}(Y_1 + Y_1^*) - \mu$$

into the first Home budget constraint, we get

$$\mu = B_2 - \frac{1}{2}(Y_1 - Y_1^*) - \frac{1}{2}(V_1 - V_1^*).$$

Substituting

$$C_2(s) = \frac{1}{2}\left[Y_2(s) + Y_2^*(s)\right] - \mu$$

into the second-period Home constraint gives

$$\mu = -(1+r)B_2.$$

(We could have gotten the same answers using Foreign's constraint, which means that Foreign's constraint holds once we find μ and B_2 such that Home's does.) Solving for μ and B_2 yields

$$\mu\left(1 + \frac{1}{1+r}\right) = \frac{1}{2}(Y_1^* - Y_1) + \frac{1}{2}(V_1^* - V_1),$$

which has the interpretation that the present discounted value of the excess of Foreign's consumption over world average consumption equals the difference between its and world average (equilibrium) date 1 resources.

(c) If $\gamma \neq \gamma^*$, a natural conjecture is that the less risk averse (lower gamma) country holds a greater share of the risky world output portfolio. Thus, for all dates/states, one might conjecture that for some μ,

$$C = \frac{\gamma^*}{\gamma + \gamma^*}Y^{\mathrm{w}} - \mu, \quad C^* = \frac{\gamma}{\gamma + \gamma^*}Y^{\mathrm{w}} + \mu,$$

by analogy with the answer to part b. To support this (plainly output-market-clearing) allocation, V_1, say, would have to satisfy the Home Euler equation

$$\exp(-\gamma C_1)V_1 = \beta \sum_s \pi(s)\, Y_2(s) \exp\left[-\gamma C_2(s)\right],$$

which it will if

$$V_1 = \beta \sum_s \pi(s)\, Y_2(s) \exp\left\{-\frac{\gamma\gamma^*}{\gamma + \gamma^*}\left[Y_2^{\mathrm{w}}(s) - Y_1^{\mathrm{w}}\right]\right\}.$$

Given this price, Foreign's Euler equation for V_1 is likewise satisfied at the "candidate" consumption allocation, as you can check. The same argument shows that

$$V_1^* = \beta \sum_s \pi(s) Y_2^*(s) \exp\left\{-\frac{\gamma\gamma^*}{\gamma+\gamma^*}\left[Y_2^{\mathrm{W}}(s) - Y_1^{\mathrm{W}}\right]\right\}$$

is consistent with the Home and Foreign Euler equations at the proposed allocation. (Set $\gamma = \gamma^*$ here to find the prices relevant for part b above.) To check budget constraints, let $\lambda \equiv \gamma^*/(\gamma + \gamma^*)$ be Home's risky portfolio share. Then calculations analogous to those in part b show that μ and B_2 satisfy Home's constraints (as well as Foreign's, by Walras's law) if

$$\mu = B_2 - \left[(1-\lambda)(Y_1 + V_1) - \lambda(Y_1^* + V_1^*)\right],$$

$$\mu = -(1+r)B_2,$$

which gives the solution for μ as

$$\mu\left(1 + \frac{1}{1+r}\right) = \frac{\gamma^*}{\gamma+\gamma^*}(Y_1^* + V_1^*) - \frac{\gamma}{\gamma+\gamma^*}(Y_1 + V_1).$$

Here, μ depends not only on relative date 1 wealth but also on risk aversion. To see how, write the preceding as

$$\mu\left(1 + \frac{1}{1+r}\right) = \frac{\gamma^*}{\gamma+\gamma^*}\left[(Y_1^* + V_1^*) - (Y_1 + V_1)\right] + \frac{\gamma^* - \gamma}{\gamma+\gamma^*}(Y_1 + V_1).$$

If the two countries have equal initial wealths, for example, the more risk averse country will have a higher deterministic consumption component.

6. (There is a mistake in the statement of the exercise. Delete the words "each period" in the third line from the bottom of the first paragraph.)

(a) Let us assume initially that risk-free bonds are indexed to the Home good (good X). (Part d below will consider the introduction of bonds indexed to good Y.) In general, the period-by-period finance constraint for the Home-country representative individual would be

$$\begin{aligned} & C_{\mathrm{X},s} + p_s C_{\mathrm{Y},s} + x_{\mathrm{X},s+1} V_{\mathrm{X},s} + x_{\mathrm{Y},s+1} V_{\mathrm{Y},s} + B_{s+1} \\ = \; & (1+r_s) B_s + x_{\mathrm{X},s}(X_s + V_{\mathrm{X},s}) + x_{\mathrm{Y},s}(p_s Y_s + V_{\mathrm{Y},s}), \end{aligned}$$

Here $x_{\mathrm{X},s}$ and $x_{\mathrm{Y},s}$ denote the fractional shares of the Home and Foreign country funds that the Home representative agent buys on date $s-1$, and r_s is the risk-free own-rate of interest on good X between dates $s-1$ and s. (Remember that $V_{\mathrm{Y},s}$ is the ex dividend date s value of the Foreign country fund measured in units of good X.) The preceding constraint looks exactly like eq. (56) in the chapter, except that we recognize the distinctness of the Home and Foreign outputs and use the relative price of Y in terms of X on date s, p_s, to express the budget constraint in terms of X. To find a representative Home agent's first-order conditions, we use the Lagrangian approach (see supplement A to Chapter 2). Form the Lagrangian:

$$\begin{aligned} \mathcal{L}_t = \mathrm{E}_t \Big\{ \textstyle\sum_{s=t}^{\infty} \beta^{s-t} \big[u(C_{\mathrm{X},s}, C_{\mathrm{Y},s}) - \lambda_s \big(C_{\mathrm{X},s} + p_s C_{\mathrm{Y},s} + x_{\mathrm{X},s+1} V_{\mathrm{X},s} + x_{\mathrm{Y},s+1} V_{\mathrm{Y},s} \\ + B_{s+1} - (1+r_s) B_s - x_{\mathrm{X},s}(X_s + V_{\mathrm{X},s}) - x_{\mathrm{Y},s}(p_s Y_s + V_{\mathrm{Y},s}) \big) \big] \Big\} . \end{aligned}$$

Differentiating with respect to $x_{\mathrm{X},t+1}$ gives the first-order condition for the Home country fund,

$$\lambda_t V_{\mathrm{X},t} = \mathrm{E}_t \left\{ \beta \lambda_{t+1} (X_{t+1} + V_{\mathrm{X},t+1}) \right\} . \tag{6}$$

(Remember that $x_{\mathrm{X},t+1}$ is a date t choice variable and that λ_t and $V_{\mathrm{X},t}$ are known to the consumer when that choice is made.) Similarly, differentiating with respect to $x_{\mathrm{Y},t+1}$ yields the optimality condition for the Foreign country fund,

$$\lambda_t V_{\mathrm{Y},t} = \mathrm{E}_t \left\{ \beta \lambda_{t+1} (p_{t+1} Y_{t+1} + V_{\mathrm{Y},t+1}) \right\} . \tag{7}$$

The condition for the riskless bond B_{t+1} is similarly derived as

$$\lambda_t = \mathrm{E}_t \left\{ \beta \lambda_{t+1} (1 + r_{t+1}) \right\} .$$

Finally, we can determine λ_t by differentiation with respect to $C_{\text{X},s}$ and $C_{\text{Y},s}$, which reveals that the Lagrange multiplier is simply the marginal utility of the Home goods:

$$\frac{\partial u(C_{\text{X},t}, C_{\text{Y},t})}{\partial X} = \lambda_t = \frac{1}{p_t} \cdot \frac{\partial u(C_{\text{X},t}, C_{\text{Y},t})}{\partial Y}. \tag{8}$$

(b) If Home and Foreign agents start with perfectly pooled, identical portfolios of risky claims, they will keep these portfolios and equal wealth levels forever. The reason is that under the initial allocation assumed, Home and Foreign agents are identical, not only ex ante but also ex post. Thus, unexpected shocks affect them equally, open up no opportunities for trade, and do not redistribute wealth between them. [If one did not start from such a perfectly pooled equilibrium, there would be no guarantee of reaching it, of course. The Lucas (1982) model is silent on how this assumed equilibrium is reached, whereas the model in section 5.3 does not require initial perfect pooling.] Equation (8) above shows that in the perfectly pooled equilibrium,

$$p_t = \frac{\partial u\left(\frac{1}{2}X_t, \frac{1}{2}Y_t\right)/\partial Y}{\partial u\left(\frac{1}{2}X_t, \frac{1}{2}Y_t\right)/\partial X}.$$

(c) Applying iterative forward substitution to eqs. (6) and (7) above, coupled with a no speculative bubbles condition, we derive

$$V_{\text{X},t} = \mathrm{E}_t\left\{\sum_{s=t+1}^{\infty} \beta^{s-t}\left[\frac{\partial u\left(\frac{1}{2}X_s, \frac{1}{2}Y_s\right)/\partial X}{\partial u\left(\frac{1}{2}X_t, \frac{1}{2}Y_t\right)/\partial X}\right] X_s\right\}.$$

In addition,

$$\begin{aligned} V_{\text{Y},t} &= \mathrm{E}_t\left\{\sum_{s=t+1}^{\infty} \beta^{s-t}\left[\frac{\partial u\left(\frac{1}{2}X_s, \frac{1}{2}Y_s\right)/\partial X}{\partial u\left(\frac{1}{2}X_t, \frac{1}{2}Y_t\right)/\partial X}\right] p_s Y_s\right\} \\ &= p_t \mathrm{E}_t\left\{\sum_{s=t+1}^{\infty} \beta^{s-t}\left[\frac{\partial u\left(\frac{1}{2}X_s, \frac{1}{2}Y_s\right)/\partial Y}{\partial u\left(\frac{1}{2}X_t, \frac{1}{2}Y_t\right)/\partial Y}\right] Y_s\right\}, \end{aligned}$$

where the two alternative ways of expressing $V_{\mathrm{Y},t}$ are derived using the expression for p from part b.

(d) An Euler-equation argument shows that the equilibrium date t price of a unit of X to be delivered with certainty on date $t+1$, $p_{\mathrm{X},t}$, is

$$p_{\mathrm{X},t} = \beta \mathrm{E}_t \left\{ \frac{\partial u\left(\frac{1}{2}X_{t+1}, \frac{1}{2}Y_{t+1}\right)/\partial X}{\partial u\left(\frac{1}{2}X_t, \frac{1}{2}Y_t\right)/\partial X} \right\}.$$

The Euler equation for the risk-free bond, derived in part a above, shows that

$$p_{\mathrm{X},t} = \frac{1}{1+r_{t+1}}.$$

Analogously, the date t price (in terms of good X) of a unit of Y to be delivered with certainty on date $t+1$, $p_{\mathrm{Y},t}$, is

$$p_{\mathrm{Y},t} = \beta \mathrm{E}_t \left\{ \frac{\partial u\left(\frac{1}{2}X_{t+1}, \frac{1}{2}Y_{t+1}\right)/\partial X}{\partial u\left(\frac{1}{2}X_t, \frac{1}{2}Y_t\right)/\partial X} \cdot p_{t+1} \right\}.$$

The price of the same security in terms good Y on date t is

$$\frac{p_{\mathrm{Y},t}}{p_t} = \beta \mathrm{E}_t \left\{ \frac{\partial u\left(\frac{1}{2}X_{t+1}, \frac{1}{2}Y_{t+1}\right)/\partial Y}{\partial u\left(\frac{1}{2}X_t, \frac{1}{2}Y_t\right)/\partial Y} \right\}.$$

Notice that the equation

$$\frac{p_{\mathrm{Y},t}}{p_t} = \frac{1}{1+r^{\mathrm{Y}}_{t+1}}$$

defines the own rate of interest on good Y between dates t and $t+1$. If we had explicitly introduced risk-free bonds denominated in good Y into the budget constraint of part a, we would have found the additional Euler equation for those bonds,

$$\frac{\partial u\left(C_{\mathrm{X},t}, C_{\mathrm{Y},t}\right)}{\partial Y} = \left(1+r^{\mathrm{Y}}_{t+1}\right)\beta \mathrm{E}_t \left\{ \frac{\partial u\left(C_{\mathrm{X},t+1}, C_{\mathrm{Y},t+1}\right)}{\partial Y} \right\},$$

which is equivalent to

$$\frac{\partial u\left(C_{\mathrm{X},t}, C_{\mathrm{Y},t}\right)}{\partial X} = \beta \mathrm{E}_t \left\{ \frac{\left(1 + r^{\mathrm{Y}}_{t+1}\right) p_{t+1}}{p_t} \cdot \frac{\partial u\left(C_{\mathrm{X},t+1}, C_{\mathrm{Y},t+1}\right)}{\partial X} \right\}.$$

This equation merely establishes the relationship between the own-rates of interest on the two goods; nothing in our analysis is changed, since bond markets actually are redundant in this model (they are never used in equilibrium). Notice that a more plausible choice for "the" riskless bond might be one with a face value indexed in some way to utility. When the period utility function takes the form

$$u(C_{\mathrm{X}}, C_{\mathrm{Y}}) = \tilde{u}\left[\Omega(C_{\mathrm{X}}, C_{\mathrm{Y}})\right] = \tilde{u}(C),$$

as in Chapter 4 [with $\Omega(C_{\mathrm{X}}, C_{\mathrm{Y}})$ homogeneous of degree one], then we can define a bond that is indexed to real consumption C. The rate of interest on that bond defines the own rate of interest on the real consumption basket C, according to the Euler equation

$$\tilde{u}'(C_t) = (1 + r^{\mathrm{C}}_{t+1})\beta \mathrm{E}_t \left\{\tilde{u}'(C_{t+1})\right\}.$$

In equilibrium, that interest rate is given by

$$\frac{1}{1 + r^{\mathrm{C}}_{t+1}} = \frac{\beta \mathrm{E}_t \left\{ \tilde{u}'\left[\Omega\left(\frac{1}{2} X_{t+1}, \frac{1}{2} Y_{t+1}\right)\right]\right\}}{\tilde{u}'\left[\Omega\left(\frac{1}{2} X_t, \frac{1}{2} Y_t\right)\right]}.$$